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Instability of viscous axial flow in annuli having a rotating inner cylinder

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Hot-wire measurements are presented of the onset of instability in developed axial and tangential flow due to inner-cylinder rotation in annuli of radius ratios 0·9, 0·81 and 0·58 for axial-flow Reynolds numbers between 86 and 2000. Measurements of the minimum critical Taylor number are reported along three radii at azimuthal locations 90° apart. At the largest radius ratio these azimuthal measurements show considerable variation but the sensitivity of marginal stability to angular orientation becomes negligible at a radius ratio of 0·58, when measurements and the Galerkin predictions of Hasoon & Martin (1977) are in close accord. The greater sensitivity to orientation as the radius ratio increases appears to correlate with measured percentage circumferential variations in annulus width arising from manufacturing tolerances and non-uniform curvature of the surfaces.

1. Introduction

Fluid motion through an annulus whose inner surface rotates is a frequently occurring engineering situation; the cooling of rotating electrical machinery by means of a rotor-mounted fan, the forced flow of lubricant in a journal bearing and the return flow of drilling mud between the rotating drill string and the stationary outer casing of the drills used for oil and gas exploration are but three of many applications. Knowledge of the fluid motion in such examples is needed at the design stage to predict hydrodynamic behaviour and hence torque and heat-transfer characteristics in relation to the transport properties of the working fluid. These will be influenced by the incidence of rotational instability and the development of wavy or vortex motion within the laminar tangential boundary layer to which inner-cylinder rotation gives rise. It is therefore important to be able to predict the onset of instability over fairly wide ranges of working fluids, annulus radius ratios $N = R_1/R_2$, where R_1 and R_2 are the respective radii of the inner and outer cylinders, and axial-flow Reynolds numbers $Re = 2\overline{W}(R_2 - R_1)/\nu$, where $\overline{W}$ is the mean axial velocity of the fluid and ν its kinematic viscosity.

Following the earlier linear stability analyses of Chandrasekhar (1960*a*, *b*) and DiPrima (1960), who predicted the critical Taylor number for marginal stability on the basis of axisymmetric disturbances in narrow-gap ($N \to 1{\cdot}0$) co-rotation situations of developed laminar tangential flow of a Newtonian fluid for Re below 200, where a uniform tangential velocity distribution was assumed, doubts arose about the validity of assuming an averaged axial velocity distribution. These were largely based on the fact that for $Re > 80$ DiPrima had reported a progressive divergence in the predicted critical Taylor number when he substituted a parabolic axial distribution, which in a narrow annulus approximates the exact developed axial profile and was thought to reflect better the possibility of Orr–Sommerfeld type effects. The divergence was such that with a stationary outer cylinder the predictions of

$$Ta_c \equiv 4\omega_c^2 R_1^2 (R_2 - R_1)^3 / [\nu^2 (R_1 + R_2)]$$

(where ω_c is the critical angular velocity of the inner cylinder) for a parabolic and an averaged axial distribution at $Re = 120$ were 15 126 and 11 850 respectively.

Despite the measurements of Donnelly & Fultz (1960) and Snyder (1962) for $Re \leqslant 200$ in an annulus with $N = 0{\cdot}95$, which generally supported the smaller Ta_c predictions of DiPrima for an averaged axial distribution, the uncertainties tended to be reinforced by the subsequent narrow-gap analyses of Hughes & Reid (1968) for large Re (> 200) with a uniform tangential distribution and Elliott (1973) for Re up to 200 with a linear tangential distribution. Both treatments assumed a parabolic axial distribution and resulted in predictions consistent with those of DiPrima for that distribution. Further questions then arose as to whether in fact the initial disturbances were axisymmetric, as had been assumed, and, if not, the extent to which marginal-stability calculations based on this assumption were in error.

The recent comparative finite-difference predictions of Hasoon & Martin (1977), which are based on the exact developed tangential velocity distribution and assume axisymmetric disturbances which result in Taylor-like vortices, do however indicate that for $N = 0{\cdot}9$ the averaged axial distribution yields values of Ta_c similar to those obtained using the exact developed axial distribution for Re up to 350. Both agree with the authors' Galerkin predictions for an averaged axial distribution, which extend to $Re = 2000$, and with their hot-wire measurements in an annulus with $N = 0{\cdot}9$ for $Re \leqslant 400$. Furthermore, for $Re \leqslant 200$, these all correspond closely to the narrow-gap predictions of DiPrima (1960) for an averaged axial distribution and to the measurements of Donnelly & Fultz (1960) and Snyder (1962) for $N = 0{\cdot}95$. Hasoon & Martin associate the lack of experimental support for predictions based on a parabolic axial profile with the inability of this distribution to satisfy the subcritical steady-state vorticity equation for other than a narrow gap.

Hasoon & Martin extended their marginal-stability predictions using the Galerkin approach for $Re \leqslant 2000$ to radius ratios of $0{\cdot}5$, $0{\cdot}3$ and $0{\cdot}1$. The hot-wire measurements of Kaye & Elgar (1958) for $N = 0{\cdot}82$ and the heat-transfer measurements of Becker & Kaye (1962) for $N = 0{\cdot}81$ correlate well with interpolated values of these predictions for Re up to about 250. Above this value measured critical Taylor numbers are less than those predicted and tend to become independent of Re. It therefore remains to investigate experimentally the validity of these predictions for smaller N (when the annulus gap becomes relatively large) and high Reynolds numbers, when the initial disturbances may be non-axisymmetric and the dominant mechanism of instability may be of the Tollmien–Schlichting type rather than Taylor-like vortices, and therefore not susceptible to analysis based on an average axial distribution.

This paper presents hot-wire measurements of Ta_c in developed axial and tangential flow in annuli of radius ratios $0{\cdot}9$, $0{\cdot}81$ and $0{\cdot}576$ for an overall range of Re of 86–2000. Measurements are reported along three radii at azimuthal positions 90° apart. While most significant for $N = 0{\cdot}9$, the circumferential variations in Ta_c at a given Re diminish to negligible percentages for $N = 0{\cdot}576$, when Ta_c agrees well with the interpolated Galerkin predictions of Hasoon & Martin; for none of the N values investigated does the measured rate of growth of Ta_c with Re confirm either that predicted by Hughes & Reid (1968) or that recently predicted by Chung & Astill (1977). The latter treatment is based on linear stability theory plus a shooting method for the exact developed axial and tangential velocity distributions; the theory covers initially axisymmetric disturbances only for $N = 0{\cdot}95$ and initially non-axisymmetric disturbances over the range $0{\cdot}1 \leqslant N \leqslant 0{\cdot}95$. The currently observed azimuthal variations in Ta_c appear to correspond to measured percentage variations in annulus width arising mainly from manufacturing tolerances.

2. Experimental apparatus and procedure

In the experimental rig shown diagrammatically in figure 1 and more fully described by Gravas (1976), air blown from the atmosphere is metered by a free-float rotameter *en route* to a conical chamber and thence through a multiplicity of plastic hose connexions to circumferential inlets to a cylindrical stilling chamber. This houses the upstream part of the test annulus, the outer stationary cylinder of overall length L_2 being foreshortened so that air enters the test annulus by flow reversal, having previously passed through the annular gap between the stilling-chamber wall and the outer cylinder in the opposite direction.

The inner cylinder of the test annulus comprises two axial sections. The upstream stationary approach length L_{1s}, part of which lies within the stilling chamber, is sufficient according to the criteria of Sparrow & Lin (1964) for 99 % development of the axial velocity profile for Re up to 2000. The downstream rotating section is mounted on a central steel shaft driven by an electric motor with speed control through a linkage V-belt and pulleys. Again the length L_{1r} of the rotating section is sufficient, according to the predictions of Martin & Payne (1972), for virtually full development of the tangential flow for values of Re up to the maxima indicated by the criterion in table 1, which also gives the above and other relevant dimensions for each of the three annuli investigated. After passage through the annulus the air is discharged to the atmosphere, its temperature being monitored at the rotameter exit, the stilling chamber and the annulus exit.

The outer stationary cylinder, which is mounted in a steel frame, is provided with a multiplicity of 4·5 mm diameter access holes (normally closed with epoxy-resin plugs) for hot-wire anemometer studies. These lie on a line along the cylinder at the axial locations Z (measured from the start of the downstream rotating section) given in table 2. The steel mounting frame has three adjustable threaded supporting bolts which allow (*a*) concentric positioning of the outer cylinder relative to the inner cylinder and (*b*) circumferential positioning of the line of access holes, along which the geared anemometer traversing mechanism is attached, by turning the outer cylinder with the bolts loosened. As already indicated, Ta_c measurements are reported for a horizontal annulus configuration with the access holes at the top and at 90° either side of this position. These positions are respectively designated as 0°, $-90°$ and $+90°$. The inner and outer surfaces of all test annuli were polished until smooth.

To measure marginal stability the hot-wire sensor assembly, incorporating a probe of 5 μm tungsten wire, is coupled to the traversing mechanism and introduced into an access hole selected as being sufficiently far downstream to give developed tangential

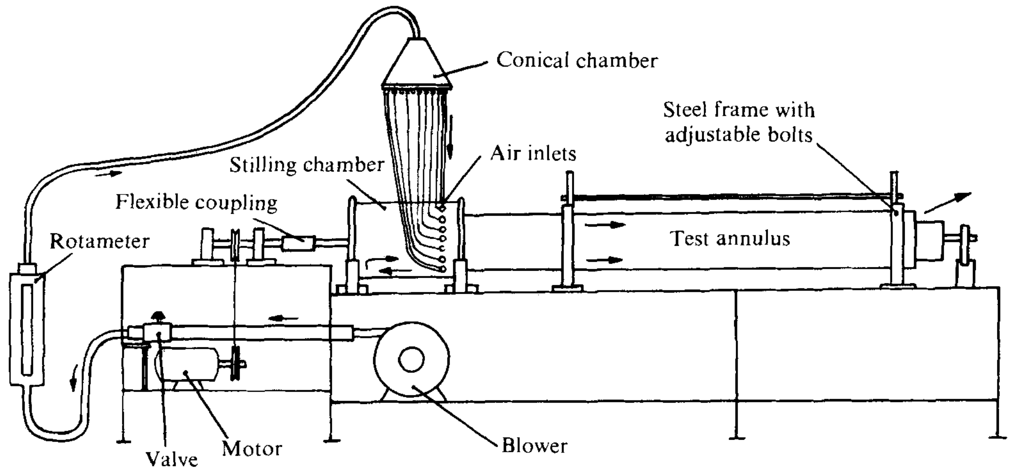

FIGURE 1. Diagram of arrangement of experimental rig.

N	0·9	0·81	0·576
R_2 (mm)	69·72	69·72	97·94
R_1 (mm)	62·75	56·49	56·49
$b = R_2 - R_1$ (mm)	6·97	13·23	41·45
L_2 (m)	2·00	2·00	2·20
L_{1s} (mm)	677	880	880
L_{1r} (m)	1·22	1·32	1·32
$Re_{max} = 12{\cdot}5L_{1r}/b$	2188	1247	398

TABLE 1. Dimensions and other details of test annuli.

N	Z (mm)													
0·9	70	125	177	227	280	330	400	485	587	692	795	895	1100	1103
0·81	60	115	168	216	268	320	370	415	575	682	755	887	990	1092
0·576	165	286	346	406	467	527	587	648	768	829	949	1010	1070	1191

TABLE 2. Axial locations of access holes (Z in mm).

flow for the chosen Re. Radial location of the sensor is achieved by optically relating the radial travel of the traversing mechanism to a tangent to a flat machined on the external surface of the outer cylinder along the line of access holes; the maximum error in radial registration does not exceed 1 % of the annular gap.

The rotational speed of the downstream section of the inner cylinder is then steadily increased until the smooth laminar trace of a strongly amplified signal on the oscilloscope, to which the anemometer output voltage is fed, abruptly gives way to a sinusoidal-like small-amplitude ripple as partly developed vortices pass the sensor. These ripples increase in intensity with further increases in rotational speed. The concept of the 'first discernible ripple' as an indicator of transition from the laminar to the laminar-plus-vortices regime was introduced by Astill (1961, 1964), who used an anemometer placed 1·27 mm from the rotating surface, but it is somewhat arbitrary in

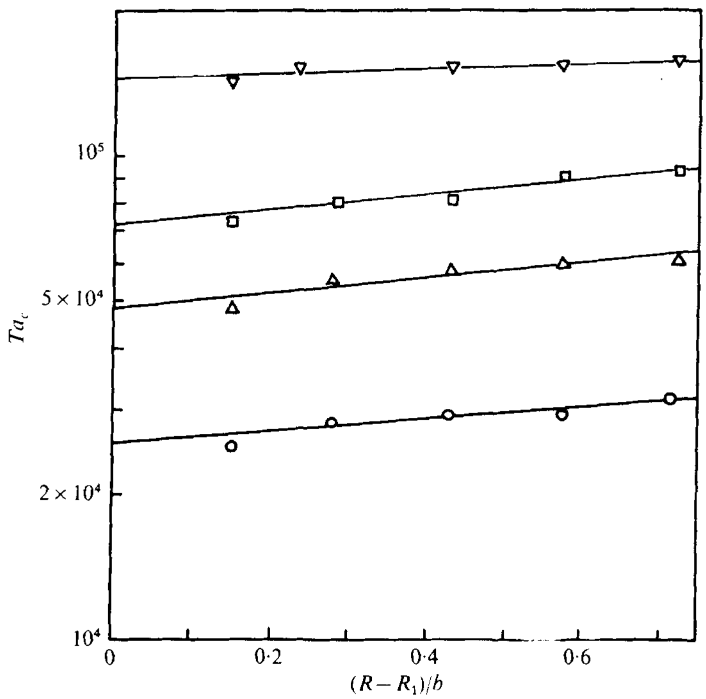

FIGURE 2. Variation of measured Ta_c for marginal stability with radial location of hot-wire sensor at $-90°$ orientation for $N = 0{\cdot}9$. ○, $Re = 150$, $2L_{1r}/(bRe) = 0{\cdot}24$; △, $Re = 300$, $2L_{1r}/(bRe) = 0{\cdot}38$; □, $Re = 500$, $2L_{1r}/(bRe) = 0{\cdot}23$; ▽, $Re = 1000$, $2L_{1r}/(bRe) = 0{\cdot}23$.

that vortex motion exhibits instability at all stages of its growth and detection depends partly on the sensitivity of the monitoring equipment. Therefore to obtain the critical Taylor number as precisely as possible measurements are made at two rotational speeds: the first when retreating from the laminar-plus-vortices regime and the second when re-entering it. The average of these speeds is found to be reproducible to within $\pm 5\,\%$. Such averages are obtained at five different radial locations; the values of Ta_c subsequently presented result from extrapolating to the rotating surface a smooth curve through these averages, as illustrated in figure 2. The entire procedure is then repeated for the two other orientations of the line of access holes.

3. Observations and comparison with predictions

Figure 2, for $N = 0{\cdot}9$, is typical of all observations in that for given Re the measured value of Ta_c falls virtually linearly with radius R as the hot-wire sensor approaches the rotating surface. Though not specifically reported by Hasoon & Martin (1977), this trend was also observed in their measurements for the same N of 0·9 but a gap width 82 % greater. It has therefore been established that the measurements provide no evidence of the initiation of hydrodynamic instability within the flow, i.e. at R other than R_1, as the Taylor number is increased from low values. The measured variations of Ta_c for each of the three orientations studied are illustrated together with the observations of Hasoon & Martin in figure 3. The latter accord well with the present measurements for the 0° orientation; those for $-90°$ and $+90°$ appear to suggest a

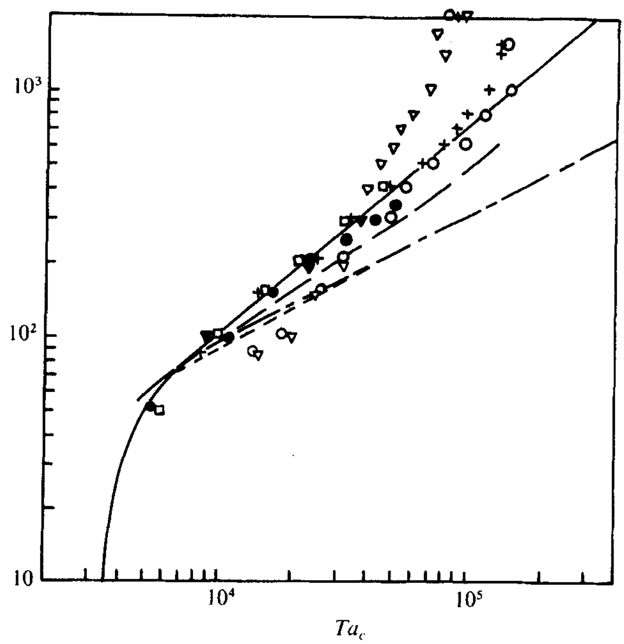

FIGURE 3. Comparison of present measurements of Ta_c for $N = 0{\cdot}9$ with earlier measurements and predictions. +, ▽, ○, current measurements for 0°, +90°, −90° respectively; □, measurements of Hasoon & Martin (1977) for $N = 0{\cdot}9$; ——, Galerkin prediction of Hasoon & Martin for $N = 0{\cdot}9$; ●, ▼, finite-difference predictions of Hasoon & Martin for $N = 0{\cdot}893$ based on average and exact axial distributions respectively; – — –, narrow-gap prediction of Hughes & Reid (1968) for a parabolic axial distribution; ‐‐‐, ———, axisymmetric and non-axisymmetric predictions respectively of Chung & Astill (1977) for $N = 0{\cdot}9$.

non-axisymmetric (and presumably sinusoidal) mode of instability for $86 \leqslant Re \leqslant 250$, with Ta_c up to 70 % greater than for 0° at a given Re, though growing at about the same rate. As Re is further increased the 0° and −90° measurements approach each other while those for +90° become progressively less than those for either 0° or −90°. The reductions in the growth rate of Ta_c with Re for both 0° and +90° at the larger values of Re are similar to those observed in the same Re range by Kaye & Elgar (1958) for $N = 0{\cdot}82$ and Becker & Kaye (1962) for $N = 0{\cdot}81$. The maximum variation in Ta_c, corresponding to a change in orientation from +90° to −90°, is 93 %, at $Re = 1000$.

Figure 3 also indicates the agreement up to $Re = 650$ between present measurements of Ta_c at 0° and the Galerkin predictions of Hasoon & Martin (1977) for an averaged axial profile at $N = 0{\cdot}9$. These are consistent with their finite-difference predictions for both the averaged and the exact axial profiles at $N = 0{\cdot}893$ over the investigated range of Re up to 350. It is further evident from figure 3 that for $80 \leqslant Re \leqslant 600$ Chung & Astill's (1977) Galerkin predictions for initially axisymmetric disturbances (scaled from $N = 0{\cdot}95$ to $0{\cdot}9$), which are based on the exact axial and tangential profiles, agree best with those of Hughes & Reid (1968) for a narrow gap and a parabolic axial distribution. Neither set of predictions is confirmed experimentally. For initially non-axisymmetric disturbances, Chung & Astill predict for $Re > 80$, and N interpolated to $0{\cdot}9$, a Ta_c–Re relationship intermediate between their own for axisymmetric disturbances and Hasoon & Martin's Galerkin predictions and measurements. The current

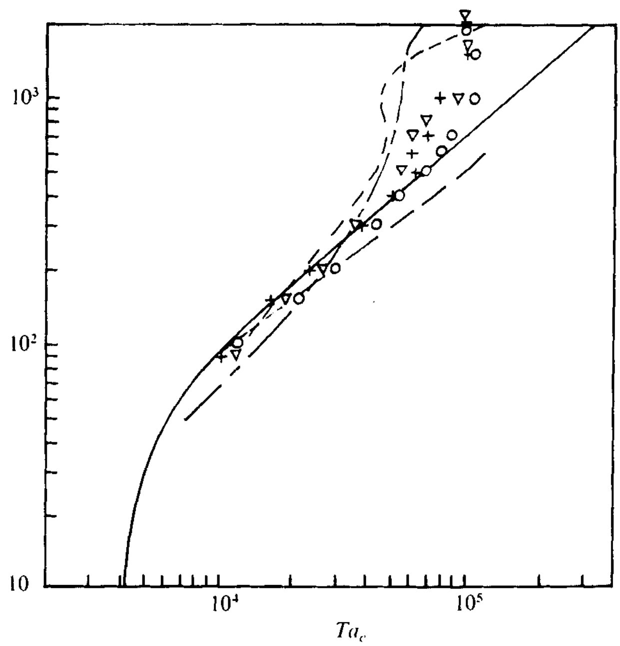

FIGURE 4. Comparison of present measurements of Ta_c for $N = 0{\cdot}81$ with earlier measurements and predictions. $+$, ∇, $\bigcirc$, current measurements for 0°, + 90°, − 90° respectively; ---, measurements of Kaye & Elgar (1958) for $N = 0{\cdot}82$; — —, measurements of Becker & Kaye (1962) for $N = 0{\cdot}81$; ——, Galerkin prediction of Hasoon & Martin (1977) for $N = 0{\cdot}81$; — — —, non-axisymmetric prediction of Chung & Astill (1977) for $N = 0{\cdot}81$.

measurements are generally overpredicted to an extent which increases with Re by Chung & Astill's calculations for both axisymmetric and non-axisymmetric disturbances. However, these measurements do appear to agree with other predictions and measurements for $Re < 80$.

The reduced effects of orientation on Ta_c at a radius ratio of 0·81 are presented in figure 4. For $92 \leqslant Re \leqslant 300$, measurements of Ta_c at − 90° and + 90° appear to suggest the same non-axisymmetric form of instability as for $N = 0{\cdot}9$, though the variation of Ta_c at a given Re is in this case little more than 30 %. As before, observations at 0° agree best up to $Re = 500$ with the Galerkin estimates of Hasoon & Martin interpolated to $N = 0{\cdot}81$; with a further increase in Re the latter progressively overpredict the current measurements, as does the Galerkin analysis for non-axisymmetric disturbances of Chung & Astill for $Re > 200$, but to a greater extent.

Within the limits of experimental uncertainty, the present measurements of Ta_c in figure 4 are comparable to those of Becker & Kaye (1962) and, in terms of trends, those of Kaye & Elgar (1958) for virtually the same N. Certain of the discrepancies between these measurements may be attributable to the fact that, as suggested by Hasoon & Martin, Kaye & Elgar's hot-wire measurements relate to axially developed but tangentially developing flow for $Re > 700$ when assessed with reference to the entry-length criteria of Sparrow & Lin (1964) and Martin & Payne (1972).

The measured variation of Ta_c with Re when $N = 0{\cdot}576$ is illustrated in figure 5 for $140 \leqslant Re \leqslant 510$. The effect of orientation on the critical Taylor number is at most 10 % and the current observations well confirm Hasoon & Martin's interpolated Galerkin

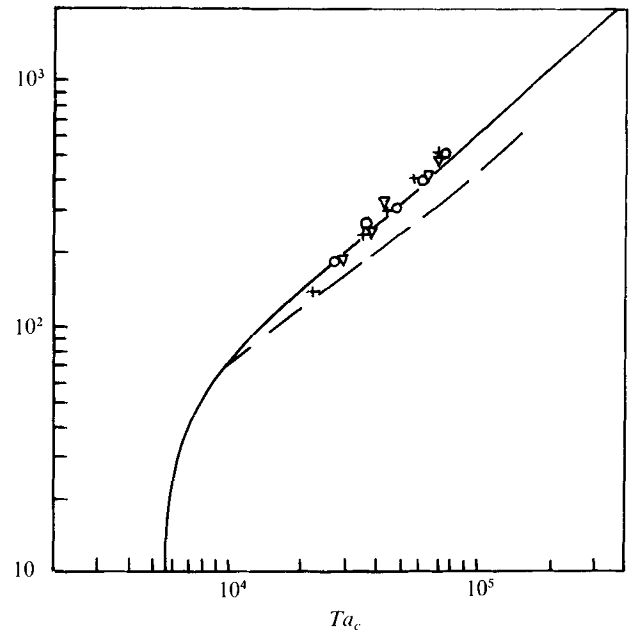

FIGURE 5. Comparison of present measurements of Ta_c for $N = 0{\cdot}576$ with earlier predictions. +, ▽, ○, current measurements for 0°, +90°, −90° respectively; ——, Galerkin prediction of Hasoon & Martin (1977) for $N = 0{\cdot}576$; — — —, non-axisymmetric prediction of Chung & Astill (1977) for $N = 0{\cdot}576$.

predictions for $N = 0{\cdot}576$. The predictions of Chung & Astill for non-axisymmetric initial disturbances are increasingly at variance with the present measurements when Re exceeds 150 at what is believed to be the smallest radius ratio for which Ta_c measurements have yet been reported in the literature. At least up to $Re = 500$ measurements for this value of N do not suggest the marked changes in the growth rate of Ta_c evident in figure 3 and 4 at the higher values of Re, which no existing theoretical treatment satisfactorily predicts.

It remains to account for the decreasing sensitivity of Ta_c to circumferential orientation with diminishing N and increasing gap size, particularly bearing in mind the disparity with Chung & Astill's predictions for non-axisymmetric initial disturbances under these circumstances. As already noted, laminar instability first occurs close to the surface of the inner cylinder and subsequently spreads radially outwards across the gap as the rotational speed is increased. It is believed that the observed orientation effect on Ta_c is attributable less to naturally occurring non-axisymmetric disturbances than to circumferential and axial variations in gap width; these variations are determined by the manufacturing tolerances on the cylinders comprising the annuli and their concentricity or otherwise when assembled. This is especially true of long cylinders of dense material, which may deflect significantly under their own weight or, in the case of the inner cylinder, with rotation. There may also be sufficient variation in the wall thickness of the stationary outer cylinder to cause misalignment with the inner cylinder especially when, as in the present experiments, the former is rotated for purposes of orientation.

While axial and circumferential variations in the diameters of the turned and polished inner cylinders already described do not exceed 0·06 %, the larger tolerances on the commercial tubes used to form the stationary outer cylinders lead to maximum measured percentage variations in b (with the inner cylinder stationary) along a 1 m length of 7·1, 4·0 and 1·6 for $N = 0{\cdot}9$, 0·81 and 0·576, respectively, for all three orientations. Since to a good approximation $Ta \propto b^3$ for a given ω, R_1 and ν, the possible percentage variations in Ta (and hence in values of Ta_c based on the measured ω_c and the data in table 1) are 22·8, 12·5 and 4·9 respectively. Qualitatively the trend of these percentages accords with the diminishing effect of orientation with decreasing N as indicated earlier in figures 3, 4 and 5. A less significant but further modifying factor is the accompanying percentage change in the assessed value of Re owing to variation in b. The fact that the measured onset of instability for the largest gap appears to be virtually axisymmetric and in accord with the Galerkin predictions of Hasoon & Martin based on this assumption and an averaged axial velocity for Re up to 510 would seem to support the above hypothesis, though further measurements are needed by way of confirmation, particularly if the information is to be useful to the designer of engineering equipment.

4. Conclusions

The following conclusions may be drawn from the foregoing.

(*a*) Over the ranges investigated, the measurements made indicate that, irrespective of the azimuthal orientation, annulus radius ratio and axial-flow Reynolds number, marginal instability first occurs close to the rotating inner cylinder and subsequently spreads radially outwards with further increases in the Taylor number.

(*b*) At the smaller values of Re the rate of growth of Ta_c and its minimum values are well predicted by the linear stability theory of Hasoon & Martin; for a given Re the critical Taylor number is sensitive to orientation to an extent which diminishes rapidly with decreasing radius ratio.

(*c*) At larger Re the variations in Ta_c for all orientations exhibit similar characteristics to those observed by Becker & Kaye and Kaye & Elgar of reduced dependence on Re.

(*d*) The predictions of Hughes & Reid and Chung & Astill are not supported by present measurements of marginal instability.

(*e*) The diminishing effect of orientation on measured values of Ta_c with decreasing radius ratio correlates with correspondingly smaller percentage variations in gap width attributable to manufacturing tolerances.

(*f*) The extent to which such imperfections determine the apparently non-axisymmetric modes of instability for the larger radius ratios is unknown; measurements for the smallest radius ratio investigated, which support the Galerkin predictions of Hasoon & Martin, nevertheless indicate that the minimum Ta_c is little influenced by such non-axisymmetric instabilities as occur when percentage variations in gap width are minimal.

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